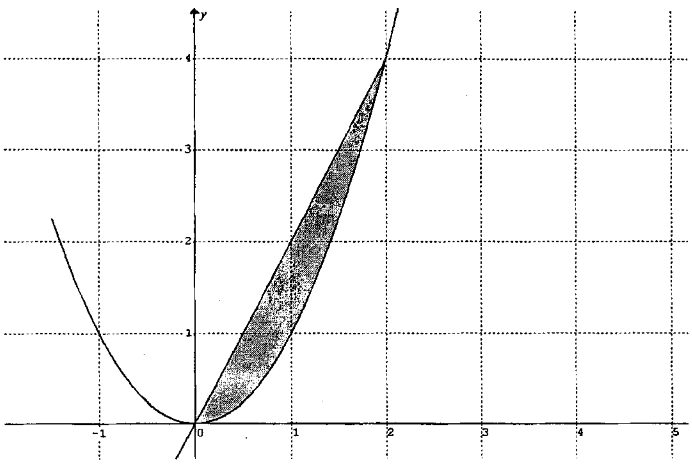
C3141

C3.Q104.NOTES: 15A DOUBLE INTEGRALS

LESSON 1 (15.1 – 15.3)

WARM UP: Find the area of the region R bounded by $y = x^2$ and $y = 2x$



$$A = \int_0^2 (2x - x^2)\,dx$$
$$= \left[x^2 - \tfrac{1}{3}x^3\right]_0^2$$
$$= 2^2 - \tfrac{1}{3}\cdot 2^3 = \frac{4}{3}$$

1. Describe some possible meanings of the double integral: $\iint\limits_R f(x,y)dA$

$f(x,y)$ = density $\underline{\text{then}}$ $\iint f(x,y)\,dA$ = mass of 2D plate with shape R

$f(x,y)$ = height/depth $\underline{\text{then}}$ $\iint f(x,y)\,dA$ = volume of solid with base/surface R

$f(x,y) = 1$ $\underline{\text{then}}$ $\iint\limits_R f(x,y)\,dA = \iint\limits_R A$ = area of a 2D region R

2. Express $\iint\limits_R f(x,y)dA$ as $\iint\limits_R f(x,y)dydx$ and $\iint\limits_R f(x,y)dxdy$

Note that: $\boxed{dA}\,dy$, dx; $dA = dx\,dy$

Fix the x $\quad \underbrace{\int_0^2 \underbrace{\int_{x^2}^{2x} f(x,y)\,dy}_{\text{inner}}\,dx}_{\text{outer}} = \int_0^4 \int_{y/2}^{\sqrt{y}} f(x,y)\,dx\,dy$

3. Evaluate $\iint\limits_R f(x,y)dA$ for $f(x,y) = 1$

$$A = \int_0^2 \underbrace{\int_{x^2}^{2x} dy}_{\text{evaluate first}}\,dx = \int_0^2 [y]_{x^2}^{2x}\,dx = \int_0^2 (2x - x^2)\,dx = \frac{4}{3} \quad (\underline{\text{Type I}})$$

FTC 2 $\qquad$ same integral as before

$\underline{\text{or}}$ $$A = \int_0^4 \int_{y/2}^{\sqrt{y}} dx\,dy = \int_0^4 [x]_{y/2}^{\sqrt{y}}\,dy = \int_0^4 (\sqrt{y} - y/2)\,dy = \frac{4}{3} \quad (\underline{\text{Type II}})$$

4. Evaluate $\iint_R f(x,y)dA$ for $f(x,y) = x^3 + 4y$

$$\iint_R f(x,y)\,dA = \int_0^2\int_{x^2}^{2x} (x^3 + 4y)\,dy\,dx$$

$$= \int_0^2 \left[x^3y + 2y^2\right]_{x^2}^{2x} dx$$

$$= \int_0^2 \left[(x^3(2x) + 2(2x)^2) - (x^3(x^2) + 2(x^2)^2\right] dx$$

$$= \int_0^2 \left(2x^4 + 8x^2 - x^5 - 2x^4\right) dx = \int_0^2 (8x^2 - x^5)\,dx$$

$$= \left[\frac{8}{3}x^3 - \frac{1}{6}x^6\right]_0^2 = \frac{64}{3} - \frac{64}{6} = \boxed{\frac{32}{3}}$$

$$\iint_R f(x,y)\,dA = \int_0^4\int_{y/2}^{\sqrt{y}} (x^3 + 4y)\,dx\,dy$$

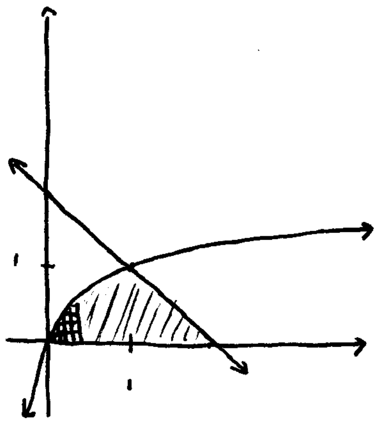
$$= \int_0^4 \left[\frac{1}{4}x^4 + 4xy\right]_{y/2}^{\sqrt{y}} dy$$

$$= \int_0^2 \left[\left(\frac{1}{4}\sqrt{y}^4 + 4\sqrt{y}\,y\right) - \left(\frac{1}{4}\cdot\left(\frac{y}{2}\right)^4 + 4\left(\frac{y}{2}\right)y\right)\right] dy$$

$$= \int_0^2 \left(\frac{1}{4}y^2 + 4y^{3/2} - \frac{1}{64}y^4 - 2y^2\right) dy = \int_0^2 \left(4y^{3/2} - \frac{1}{64}y^4 - \frac{7}{4}y^2\right) dy$$

5. Express $\iint_R f(x,y)\,dA$ as $\iint_R f(x,y)\,dy\,dx$ and $\iint_R f(x,y)\,dx\,dy$ for

R = Region bounded by: $x = y^3$, $x + y = 2$, $y = 0$



$$\iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy$$

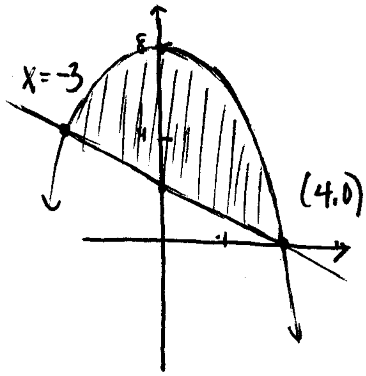
$$= \int_0^1 \int_{y^3}^{2-y} f(x,y)\,dx\,dy$$

$$\text{OR} = \int_0^1 \int_0^{\sqrt[3]{x}} f(x,y)\,dy\,dx + \int_1^2 \int_0^{2-x} f(x,y)\,dy\,dx$$

6. Express $\iint_R f(x,y)dA$ as $\iint_R f(x,y)dydx$ **OR** $\iint_R f(x,y)dxdy$

R = Region bounded by: $2y = 16 - x^2$, $x + 2y = 4$ $\rightarrow$ $y = -\frac{1}{2}x + 2$

$y = 8 - \frac{1}{2}x^2$



Fix the x $\longrightarrow$ Type I (II not possible)

$$\iint_R f(x,y)\,dA = \int_{-3}^{4}\int_{(-\frac{1}{2}x+2)}^{(8-\frac{1}{2}x^2)} f(x,y)\,dy\,dx$$

7. Draw the region R in $\int_{1}^{3}\int_{\pi/6}^{y^2} f(x,y)dxdy$. Evaluate $\int_{1}^{3}\int_{\pi/6}^{y^2} f(x,y)dxdy$ for $f(x,y)=2y\cos(x)$

$$\int_{1}^{3}\int_{\pi/6}^{y^2} (2y\cos x)\,dx\,dy$$

$$= \int_{1}^{3} \left[2y\sin x\right]_{\pi/6}^{y^2} dy$$

$$= \int_{1}^{3} \left(2y\sin(y^2) - 2y\sin\tfrac{\pi}{6}\right) dy$$

$$= \int_{1}^{3} \left(2y\sin(y^2) - y\right) dy$$

$$= \left[-\cos(y^2) - \tfrac{1}{2}y^2\right]_{1}^{3}$$

$$= \left(-\cos(9) - \tfrac{9}{2}\right) - \left(-\cos(1) - \tfrac{1}{2}\right) = \boxed{\cos 1 - \cos 9 - 4}$$

8. Evaluate $\iint_R f(x,y)dA$ where R is the rectangular region $\overset{x}{[1,4]}\times\overset{y}{[-1,2]}$ and $f(x,y)=2x+6x^2y$

$$\iint_R f(x,y)\,dA = \overbrace{\int_{-1}^{2}\overbrace{\int_{1}^{4}(2x+6x^2y)\,dx}^{x}\,dy}^{y}$$

$$= \int_{-1}^{2}\left[x^2 + 2x^3y\right]_{1}^{4} dy$$

$$= \int_{-1}^{2}\left((16+128y) - (1+2y)\right) dy = \int_{-1}^{2}(15+126y)\,dy$$

$$= \left[15y + 63y^2\right]_{-1}^{2}$$

$$= (30+252) - (-15+63) = \boxed{234}$$

C3.Q104.NOTES: 15A DOUBLE INTEGRALS

LESSON 3 (15.4, 15.9)

$T: \quad x = x(u,v) \qquad y = y(u,v)$

$$\iint_{R_{xy}} f(x,y)\,dx\,dy = \iint_{T_{uv}} f(x(u,v), y(u,v)) \underbrace{\left| \frac{\partial(x,y)}{\partial(u,v)} \right|}_{\text{Jacobian}} du\,dv$$

with Jacobian $= J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

Polar Transformation

$T: x = r\cos\theta \quad y = r\sin\theta \qquad |J| = \left| \dfrac{\partial(x,y)}{\partial(r,\theta)} \right| = r$

Therefore: $\displaystyle\iint_{R_{xy}} f(x,y)\,dx\,dy = \iint_{T_{r\theta}} f(x(r,\theta), y(r,\theta))\, r\,dr\,d\theta$

→ Rethinking u-substitutions

$$\int_x f(x)\,dx = \int f(x(u)) \frac{dx}{du} \cdot du$$

Trad'l: $\displaystyle\int \frac{1}{2x+5}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|2x+5|$

$u = 2x+5 \quad du = 2\,dx$

Transformational: $\displaystyle\int \frac{1}{2x+5}\,dx = \int \frac{1}{2\left(\frac{u-5}{2}\right)+5}\,\frac{1}{2}\,du = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|2x+5|$

$T: \ u = 2x+5$

$x = \dfrac{u-5}{2} \qquad \dfrac{dx}{du} = \dfrac{1}{2}$

1. (WARM UP) SET UP only the integral used to find the volume of the solid bounded by $z = 4 - x^2 - y^2$ and the xy-plane.

$$\iint_R (4 - x^2 - y^2)\, dx\, dy = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - x^2 - y^2)\, dy\, dx$$

2. Find the volume of the solid described in #1 using a polar transformation.

$$T: x = r\cos\theta \quad y = r\sin\theta \quad |r| = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r$$

$$\iint_{xy} (4 - (x^2+y^2))\, dy\, dx = \int_0^{2\pi}\int_0^{2} (4 - r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[2r^2 - \frac{1}{4}r^4\right]_0^2 d\theta$$

$$= \int_0^{2\pi} (8 - 4)\, d\theta = \int_0^{2\pi} 4\, d\theta = [4\theta]_0^{2\pi} = \boxed{8\pi}$$

$$\rightarrow \text{OR } V = \int_0^{2\pi} d\theta \int_0^{2} (4 - r^2)\, r\, dr = 2\pi \cdot 4 = \boxed{8\pi}$$

3. Evaluate $\iint_{R_{xy}} e^{-(x^2+y^2)} dx dy$ for the region R bounded in the first quadrant by the circles $x^2 + y^2 = 1$ $x^2 + y^2 = 4$.

$$\iint_{R_{xy}} e^{-(x^2+y^2)}\, dx\, dy = \int_0^{\pi/2}\int_1^{2} e^{-r^2}\, r\, dr\, d\theta \qquad u = -r^2 \quad du = -2r\, dr$$

$$= \int_0^{\pi/2} -\frac{1}{2}\int_1^{2} e^{u}\, du\, d\theta$$

$$= \int_0^{\pi/2} -\frac{1}{2}\left[e^{u}\right]_{-1}^{-4} d\theta = \int_0^{\pi/2} -\frac{1}{2}(e^{-4} - e^{-1})\, d\theta$$

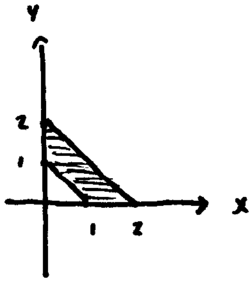
$$= -\frac{1}{2}(e^{-4} - e^{-1})\, [\theta]_0^{\pi/2} = \boxed{-\frac{\pi}{4}(e^{-4} - e^{-1})}$$

4. Show that the transformation $T: x = r\cos\theta,\ y = r\sin\theta$ always yields $|J| = r$.

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r \qquad |J| = r \quad \text{QED.}$$

5. Evaluate $\iint\limits_{R_{xy}} e^{(y-x)/(y+x)}dxdy$ where R is the region within the trapezoid defined by the points $(0,1)\,(0,2)\,(2,0)\,(1,0)$.



<u>Seek</u>: $x = x(u,v)$, $y = y(u,v)$

<u>Strategy</u>: $\left.\begin{array}{l} u = y - x \\ v = y + x \end{array}\right\} \rightarrow \begin{array}{l} y = \frac{1}{2}(u+v) \\ x = \frac{1}{2}(v-u) \end{array}$

$T: \ x = \frac{1}{2}(v-u) \quad y = \frac{1}{2}(u+v)$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{2}$$

THIS is the Jacobian

$|J| = \frac{1}{2} \leftarrow$ THIS is what we plug in

<u>Solve</u>: $\iint\limits_{R_{xy}} e^{\frac{y-x}{y+x}}\,dy\,dx = \iint\limits_{R_{uv}} e^{u/v} \cdot \frac{1}{2}\,du\,dv$

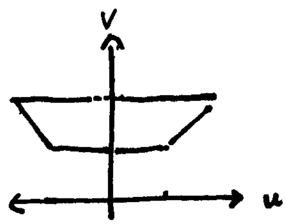
<u>Paths</u>: Write out the paths — $x+y=1 \quad x+y=2 \quad x=0 \quad y=0$

Convert to u and v — $x+y = \frac{1}{2}(u+v) + \frac{1}{2}(v-u) = v$

$x = \frac{1}{2}(u+v) \rightarrow u = -v$

$y = \frac{1}{2}(v-u) \rightarrow u = v$

So:
$x+y = 1 \rightarrow v = 1$
$x+y = 2 \rightarrow v = 2$
$x = 0 \rightarrow u = -v$
$y = 0 \rightarrow u = v$



<u>NOW Solve</u>: $\displaystyle\int_1^2\int_{-v}^{v} \frac{1}{2}e^{u/v}\,du\,dv$

$$= \int_1^2 \frac{1}{2}\left[ve^{u/v}\right]_{-v}^{v} dv = \int_1^2 \frac{1}{2}\left(ve - ve^{-1}\right) dv$$

$$= \frac{1}{2}(e - e^{-1})\int_1^2 v\,dv = \frac{1}{2}(e-e^{-1}) \cdot \frac{1}{2}\left[v^2\right]_1^2$$

$$= \boxed{\frac{3}{4}(e-e^{-1})}$$

C3143

The Statistics Connection

Let x : # of TVs in a house

$f(x)$ = frequency = probability density function

x	0	1	2	3	4	5	
$f(x)$ (freq.)	1	2	5	8	3	1	$n = 20$
$f(x)$ (pdf)	$1/20$	$2/20$	$5/20$	$8/20$	$3/20$	$1/20$	

Mean $\bar{x} = \dfrac{\sum x \cdot f(x)_{freq.}}{\sum f(x)_{freq.}} = 3.25 \Rightarrow$ average # of TVs

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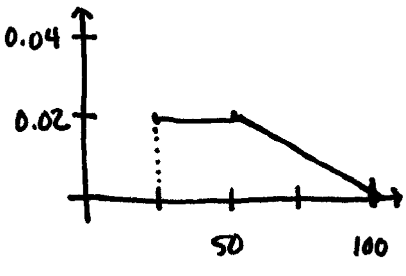
Let x = mass in dgrams of cupcake

$f(x)$ = pdf (continuous) of cupcake masses

$$f(x) = \begin{cases} 0.02 & 25 \le x \le 50 \\ 0.04 - 0.0004x & 50 \le x \le 100 \\ 0 & \text{else} \end{cases}$$



Verify valid pdf:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{100} f(x)\,dx = \text{Area} = 1$$

Average x :

$$\bar{x} = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_{25}^{100} x \cdot f(x)\,dx =$$

$$= \int_{25}^{50} 0.02x\,dx + \int_{50}^{100} (0.04 - 0.0004x)\,x\,dx$$

$$= 52.08 \text{ dg}$$

Now in multivariable:

$$\bar{x} = \frac{\int_a^b x\, f(x)\, dx}{\int_a^b f(x)\, dx} \Rightarrow \bar{x} = \frac{\iint_{xy} x \cdot f(x,y)\, dA}{\iint_{xy} f(x,y)\, dA}$$

(2D)

$$\bar{y} = \frac{\iint_{xy} y \cdot f(x,y)\, dA}{\iint_{xy} f(x,y)\, dA}$$

Center of Mass $(\bar{x},\bar{y})$

Average x not to be confused with average f(x):

$$\text{Av } f(x,y) = \frac{\iint f(x,y)\, dA}{\underbrace{\iint dA}_{\text{area}}}$$

ex: density, avg. temperature

Mass functions usually given as $\rho(x,y)$:

- prop. to distance from origin:

$$\rho(x,y) = k\sqrt{x^2+y^2}$$

- prop to distance from x-axis:

$$\rho(x,y) = k|y| = k\sqrt{y^2}$$